INDIAN STATISTICAL INSTITUTE
Semester Examination, Second Semester 2020-2021
Algebraic Geometry, M.Math II nd year
Total Marks 70.

k is an algebraically closed field and by a variety we mean a quasi-projective k-variety.

1 Basic Theory

1. Find the multiple points, and the tangent lines at the multiple points, for each of the following curves:

   (a) $Y^3 - Y^2 + X^3 - X^2 + 3XY^2 + 3X^2Y + 2XY$; [6]

   (b) $X^4 + Y^4 - X^2Y^2$. [6]

2. Show that an irreducible curve in $\mathbb{P}^2_k$ has at most finitely many singular points. [8]

3. Show that any nonsingular cubic curve C in $\mathbb{P}^2_k$ is equivalent to the cubic defined by
$$y^2z = x(x-z)(x-\lambda z),$$
for some $\lambda \in k \setminus \{0,1\}$. [10]

4. For X and T two k-varieties, let
$$X(T) := \{f : T \to X | f \textit{ morphism}\}.$$
If $f : X \to Y$ is a morphism of varieties , define $f_T : X(T) \to Y(T)$ by the formula $f_T(g) = f \circ g$.

   (a) Show that f is an isomorphism if and only if f_T is a bijection for every variety T. [5]

   (b) Show that intersection of two affine open subvarieties U, V of a variety X is again an affine variety. [5]

   (c) Show that $f : X \to Y$ is an isomorphism if and only if f_T is a bijection for all affine varieties T. [7]

5. (Local ring of a subvariety) Let $Y \subset X$ be a subvariety.

   (a) Let $O_{Y,X}$ be the set of equivalence classes $< U, f >$ where $U \subset X$ is open with $U \cap Y$ non empty, and f is a regular function on U. We say that $< U, f > \sim < V, g >$ if $f = g$ on $U \cap V$. Show that $O_{Y,X}$ is a local ring. Show that for any irreducible curve C in $\mathbb{A}^2$, the local ring $O_{C,\mathbb{A}^2}$ is a D.V.R. [6+5]

(b) Let $\kappa(Y, X)$ be the set of equivalence classes $< U \cap Y, f >$ where $U \subset X$ is open with $U \cap Y$ non empty, and f is a regular function on $U \cap Y$. We say that $< U \cap Y, f > \sim < V \cap Y, g >$ if $f = g$ on $U \cap V \cap Y$. Show that the residue field of $O_{Y,X}$ is isomorphic to $\kappa(Y, X)$. [5]

(c) Construct an example of a two dimensional variety X and a subvariety $Y \subset X$ of dimension 1, such that $O_{Y,X}$ is not a D.V.R. [7]